# 7.4-7.7-Matrices Unit

This packet is intended to help you with your last unit of Trigonometry. You will have a test on all of this information on your last day (May 21st or 22nd).

| Topics | HW Assignment | Book Section | Packet Page Numbers |
|---|---|---|---|
| • Introduction to Matrix<br>• Adding and Subtracting Matrices<br>• Multiplying Matrices | Pg 536 #3, 7, 11, 23-37 odd | 7.5 | 1-4 |
| • Matrices in the Graphing Calculator<br>• Solving Systems Algebraically<br>• Solving Systems with Row Echelon Form | Pg 522 #41, 44, 53, 54, 55, 56, 60 (solve #55 and #56 algebraically and check with a calculator) | 7.4 | 5, 6 |
| • Determinants of a 2x2 matrix | Pg 556 #1-7 odd, 37, 38, 51-55 | 7.7 | 7 |
| • Solving Matrix Equations<br>• Matrix Inverse and Identity Matrix<br>• Solving Matrix Equations<br>• Solving Systems using Matrices (algebraically) | Pg 547 #3, 5, 11-15 odd, 49, 53, **Packet Page 10** | 7.6 | 7-8 |
| Review Assignment: Pg 572 #145, 147, 81, 84, 85, 87, 90, 91 (#85 and #90 solve algebraically and check with calc), 93-103 odd, 109-114, 121, 123, 131, 134, 145-149 odd, Pg 523 #42, 43, Pg 548 #50, 53 | | | |

# Introduction to the Matrix

A **matrix** (plural **matrices**) is sort of like a "box" of information where you are keeping track of things both right and left (**columns**), and up and down (**rows**). Usually a matrix contains **numbers** or **algebraic expressions**. You may have heard matrices called **arrays**, especially in computer science.

As an example, if you had three sisters, and you wanted an easy way to store their age and number of pairs of shoes, you could store this information in a matrix. The actual matrix is inside and includes the brackets:

$$\begin{array}{rc} & \begin{matrix} \text{Ashley} & \text{Emma} & \text{Chloe} \end{matrix} \\ \begin{matrix} \text{Age} \\ \text{Number of Pairs of Shoes} \end{matrix} & \begin{bmatrix} 23 & 18 & 15 \\ 5 & 23 & 12 \end{bmatrix} \end{array}$$

Matrices are called multi-dimensional since we have data being stored in different directions in a grid.

The **dimensions** of this matrix are "**2 x 3**" or "**2** by **3**", since we have **2** rows and **3** columns. (You always go down first, and then over to get the dimensions of the matrix).

Again, matrices are great for storing numbers and variables – and also great for solving systems of equations, which we'll see later. Each number or variable inside the matrix is called an entry or **element**, and can be identified by **subscripts**. For example, for the matrix above, "Ashley's number of pairs of shoes (**5**)" would be identified as $a_{2,1}$, since it's on the 2nd row and it's the 1st entry.

# Adding and Subtracting Matrices

Let's look at a matrix that contains numbers and see how we can **add** and **subtract** matrices.

Let's say you're in avid reader, and in June, July, and August you read fiction and non-fiction books, and magazines, both in paper copies and online. You want to keep track of how many different types of books and magazines you read, and store that information in matrices. Here is that information, and how it would look in matrix form:

**June**

| | Paper | Online |
|---|---|---|
| Fiction | 2 | 4 |
| Non-Fiction | 3 | 1 |
| Magazines | 4 | 5 |

Matrix Form:

**July**

| | Paper | Online |
|---|---|---|
| Fiction | 3 | 2 |
| Non-Fiction | 1 | 1 |
| Magazines | 5 | 3 |

Matrix Form:

**August**

| | Paper | Online |
|---|---|---|
| Fiction | 1 | 3 |
| Non-Fiction | 2 | 3 |
| Magazines | 4 | 6 |

Matrix Form:

We can **add matrices** if the dimensions are the same; since the three matrices are all "**3** by **2**", we can add them. For example, if we wanted to know the total number of each type of book/magazine we read, we could add each of the elements to get the sum:
Thus we could see that we read **6** paper fiction, **9** online fiction, **6** paper non-fiction, **5** online non-fiction books, and **13** paper and **14**online magazines.
We could also subtract matrices this same way.

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 2+3+1 & 4+2+3 \\ 3+1+2 & 1+1+3 \\ 4+5+4 & 5+3+6 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 6 & 5 \\ 13 & 14 \end{bmatrix}$$

If we wanted to see how many book and magazines we would have read in August if we had **doubled** what we actually read, we could multiply the August matrix by the number **2**. This is called **matrix scalar multiplication**; a **scalar** is just a single number that we multiply with every entry. Note that this is **not** the same as multiplying 2 matrices together (which we'll get to next):

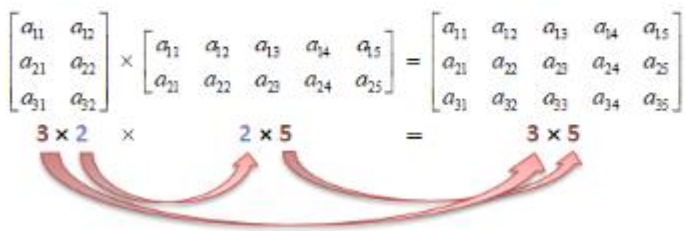
$$2\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1\times 2 & 3\times 2 \\ 2\times 2 & 3\times 2 \\ 4\times 2 & 6\times 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 6 \\ 8 & 12 \end{bmatrix}$$

# Multiplying Matrices

Multiplying matrices is a little trickier. First of all, you can only multiply matrices if the dimensions "match"; the **second dimension (columns) of the first matrix has to match the first dimension (rows) of the second matrix**, or you can't multiply them. Think of it like the **inner dimensions have to match**, and the resulting dimensions of the new matrix are the **outer dimensions**.

Here's an example of matrices with dimensions that would work:

Notice how the "middle" or "inner" dimensions of the first matrices have to be the same (in this case, "**2**"), and the new matrix has the "outside" or "outer" dimensions of the first two matrices ("**3** by **5**").




Now, let's do a **real-life example** to see how the multiplication works**.** Let's say we want to find the final grades for 3 girls, and we know what their averages are for tests, projects, homework, and quizzes. We also know that tests are **40%** of the grade, projects **15%**, homework **25%**, and quizzes **20%**.

Here's the data we have:

| Student | Tests | Projects | Homework | Quizzes |
|---|---|---|---|---|
| Alexandra | 92 | 100 | 89 | 80 |
| Megan | 72 | 85 | 80 | 75 |
| Brittney | 88 | 78 | 85 | 92 |

| Type | Weight |
|---|---|
| Tests | 40% (.4) |
| Projects | 15% (.15) |
| Homework | 25% (.25) |
| Quizzes | 20% (.2) |

Let's organize the following data into two matrices, and perform matrix multiplication to find the final grades for Alexandra, Megan, and Brittney. To do this, you have to multiply in the following way:

| Matrix | Multiplication |
|---|---|
| $\begin{bmatrix}92&100&89&80\\72&85&80&75\\88&78&85&92\end{bmatrix}\times\begin{bmatrix}.4\\.15\\.25\\.2\end{bmatrix}=\begin{bmatrix}90\\77\\87\end{bmatrix}$ | Think of turning the first matrix to the right and sideways, multiplying each number by the numbers in the second matrix, and then adding them together.<br><br>For example,<br>$(92\times.4)+(100\times.15)+(89\times.25)+(80\times.2)=90.05$<br>$(72\times.4)+(85\times.15)+(80\times.25)+(75\times.2)=76.55$<br>$(88\times.4)+(78\times.15)+(85\times.25)+(92\times.2)=86.55$ |

Just remember when you put matrices together with matrix multiplication**, the columns (what you see across) on the first matrix have to correspond to the rows down on the second matrix**. You should end up with entries that correspond with the entries of each row in the first matrix.
For example, with the problem above, the columns of the first matrix each had something to do with Tests, Projects, Homework, and Quizzes (grades). The row down on the second matrix each had something to do with the same four items (weights of grades). But then we ended up with information on the three girls (rows down on the first matrix).

**Alexandra has a 90, Megan has a 77, and Brittney has an 87**. See how cool this is?  Matrices are really useful for a lot of applications in "real life"!

Now let's do another example; let's multiply the following matrices:

$$\underset{3\times 2}{\begin{bmatrix}2&-1\\3&2\\7&5\end{bmatrix}}\times\underset{2\times 5}{\begin{bmatrix}0&-4&3&1&4\\6&7&2&9&-3\end{bmatrix}}=\underset{3\times 5}{\begin{bmatrix}-6&-15&4&-7&11\\12&2&13&21&6\\30&7&31&52&13\end{bmatrix}}$$

Note how we are turning the first matrix sideways and to the right for the operations with the second matrix. Also notice the spot where we put the "answer". For example,

$(2 \times 0) + (-1 \times 6) = -6$
$(3 \times 0) + (2 \times 6) = 12$
$(7 \times 1) + (5 \times 9) = 52$

Oh, one more thing! Remember that **multiplying matrices is not commutative** (order makes a difference), but **is associative** (you can change grouping of matrices when you multiply them). **Multiplying matrices is also distributive** (you can "push through" a matrix through parentheses), as long as the matrices have the correct dimensions to be multiplied.

# Matrices in the Graphing Calculator

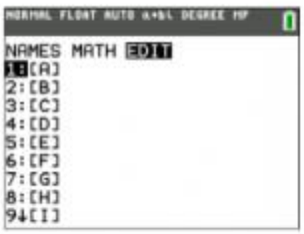
The TI graphing calculator is great for matrix operations! Here are some basic steps for storing, multiplying, adding, and subtracting matrices:

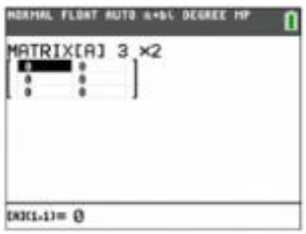
**Matrices in the Calculator – Steps and Screens**

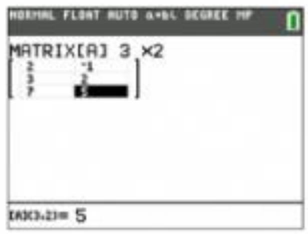
Let's multiply the following matrix using the calculator:

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 7 & 5 \end{bmatrix} \times \begin{bmatrix} 0 & -4 & 3 & 1 & 4 \\ 6 & 7 & 2 & 9 & -3 \end{bmatrix} = \begin{bmatrix} -6 & -15 & 4 & -7 & 11 \\ 12 & 2 & 13 & 21 & 6 \\ 30 & 7 & 31 & 52 & 13 \end{bmatrix}$$

To store matrices, hit , and you'll get a screen with NAMES, MATH, and EDIT, where you can identify, perform operations, and edit (store) the matrices. Hit twice so that **EDIT** is highlighted; then hit **ENTER**. Then type **3, ENTER, 2, ENTER**, (dimensions of the first matrix) and type in each value (rows at a time), followed by **ENTER**. This stores the first matrix in **[A]**:



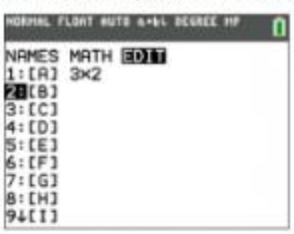



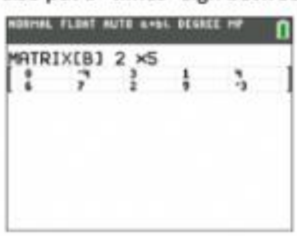

Note that you can use the cursor buttons to go back, forth, up or down to change any of the entries.

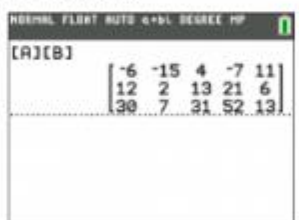
When you're done, hit to get back to the main screen.

We'll store the second matrix in **[B]** by doing the same, but after scrolling to the right to **EDIT**, scroll down to highlight **2** (for **[B]**), and hit **ENTER**. Enter the dimensions and matrix the same way. Again, key in to get back to the main screen.

To multiply matrices, hit (or **ENTER**, since the cursor is at **1**), then (or scroll down to **2** and hit **ENTER**), **ENTER**. You can also put a "**times**" sign between the matrices:



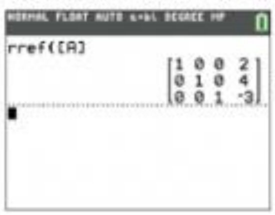




You can add and subtract matrices this same way, if the matrices have the same dimensions. You can see if you try to add or subtract these matrices, you'll get in error, since they aren't the same dimensions.

# Solving Systems with Reduced Row Echelon Form

There's another way to solve systems by converting a systems' matrix into **reduced row echelon form**, where we can put everything in one matrix (called an **augmented matrix**).
I show how to use this method by hand here in the **Solving Systems using Reduced Row Echelon Form** section, but here I'll just show you how to easy it is to solve using **RREF** in a **graphing calculator**:

| Convert System to Matrices | RREF in Calculator |
|---|---|
| $\begin{aligned} 5x-6y-7z &= 7 \\ 6x-4y+10z &= -34 \\ 2x+4y-3z &= 29 \end{aligned}$<br><br>$A \times X = B$<br>$\begin{bmatrix} 5 & -6 & -7 \\ 6 & -4 & 10 \\ 2 & 4 & -3 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -34 \\ 29 \end{bmatrix}$ | Put 3 by 4 matrix $\begin{bmatrix} 5 & -6 & -7 & 7 \\ 6 & -4 & 10 & -34 \\ 2 & 4 & -3 & 29 \end{bmatrix}$ in **[A]**.<br><br>After you've stored the square matrix, hit 2ND MATRIX $x^{-1}$, and hit ▸ once so that **MATH** is highlighted. Hit ALPHA APPS for **B** (without the ENTER), or scroll up or down to "**rref(**" and hit **ENTER.**<br><br>Then type 2ND MATRIX $x^{-1}$, and hit **ENTER** for matrix **[A]**, or scroll to the matrix you want. Then hit **ENTER** once more and you'll a matrix that looks like this:<br><br>Ignore the identity matrix; $\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$ is the answer for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. |

# Determinants, the Matrix Inverse, and the Identity Matrix

Soon we will be solving **Systems of Equations** using matrices, but we need to learn a few mechanics first!
Most **square** matrices (same dimension down and across) have what we call a **determinant**, which we'll need to get the **multiplicative inverse** of that matrix. The **inverse** of a matrix is what we multiply that square matrix by to get the **identity** matrix. We'll use the inverses of matrices to solve **Systems of Equations**; the inverses will allow us to get variables by themselves on one side (like "regular" algebra). You'll solve these mainly by using your **calculator**, but you'll also have to learn how to get them "by hand".
Note that the determinant of a matrix can be designated by $\det[A]$ or $|A|$, and the inverse of a matrix by $A^{-1}$.
Let's do some examples and first get the **determinant of matrices** (which we can get easily on a calculator!). The determinant is always just a scalar (number), and you'll see it with two lines around the matrix:

| Matrix Determinant | Notes |
|---|---|
| **2 by 2 matrix:** $\det\begin{bmatrix} 3 & 1 \\ 4 & 8 \end{bmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & 8 \end{vmatrix} = (3 \times 8) - (1 \times 4)$ $= 24 - 4 = 20$ | With a **2 by 2 matrix**, you start with the **upper left corner**, multiply diagonally down, and then **subtract** the product where you multiply down diagonally from the upper right corner. |

Now let's use the determinant to get the **inverse of a matrix**. We'll only work with **2** by **2** matrices, since you'll probably be able to use the calculator for larger matrices. Note again that **only square matrices have inverses**, but there are square matrices that don't have one (when the determinant is **0**):

| Matrix Inverse | Notes |
|---|---|
| **For Matrix A, inverse is $A^{-1}$:** $\text{Inverse}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$ **Examples:** $\text{Inverse}\begin{bmatrix} 3 & 1 \\ 4 & 8 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 8 & -1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{20} \\ -\frac{1}{5} & \frac{3}{20} \end{bmatrix}$ $\text{Inverse}\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} = \frac{1}{0}\begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} = \text{No Inverse}$ | By definition, the inverse of a matrix is the reciprocal of the determinant, multiplied by a **"switch-op"** matrix: switch the diagonals starting from the upper left, and take the opposites of the diagonals starting from the upper right. Note that when the **determinant is 0**, the reciprocal is undefined; therefore, there is **no inverse matrix.** |

Note that a matrix, multiplied by its inverse, if it's defined, will always result in what we call an **Identity Matrix**: An identity matrix has **1**'s along the diagonal starting with the upper left, and **0**'s everywhere else.

$$\begin{bmatrix} 3 & 1 \\ 4 & 8 \end{bmatrix} \times \begin{bmatrix} \frac{2}{5} & -\frac{1}{20} \\ -\frac{1}{5} & \frac{3}{20} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

# Solving Systems with Matrices

Why are we doing all this crazy math? Because we can **solve systems** with the **inverse of a matrix**, since the inverse is sort of like dividing to get the variables all by themselves on one side.
To solve systems with matrices, we use $X=A^{-1}B$. Here is why, if you're interested in the "theory" (the column on the right provides an example with "regular" multiplication). ($\boldsymbol{I}$ is the identity matrix.)

$$\begin{aligned} AX &= B & \qquad 5x &= 10 \\ A^{-1}AX &= A^{-1}B & \qquad \tfrac{1}{5}\cdot 5x &= \tfrac{1}{5}\cdot 10 \\ IX &= A^{-1}B & \qquad 1x &= 2 \\ X &= A^{-1}B & \qquad x &= 2 \end{aligned}$$

Let's take the system of equations that we worked with earlier and show that it can be solved using matrices:

| Convert System to Matrices | Solve with Matrices |
|---|---|
| $(1)x + (1)y = 6$<br>$25x + 50y = 200$<br><br>$A \times X = B$<br>$\begin{bmatrix} 1 & 1 \\ 25 & 50 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 200 \end{bmatrix}$<br><br>$\begin{bmatrix} 1 & 1 \\ 25 & 50 \end{bmatrix}$ is called the **coefficient matrix**, $\begin{bmatrix} x \\ y \end{bmatrix}$ is called the **variable matrix**, and $\begin{bmatrix} 6 \\ 200 \end{bmatrix}$ is called the **constant matrix**. | $X = A^{-1}B$<br>$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 25 & 50 \end{bmatrix}^{-1} \times \begin{bmatrix} 6 \\ 200 \end{bmatrix}$<br>$\left( \det \begin{bmatrix} 1 & 1 \\ 25 & 50 \end{bmatrix} = 25 \right)$<br>$= \frac{1}{25} \begin{bmatrix} 50 & -1 \\ -25 & 1 \end{bmatrix} \times \begin{bmatrix} 6 \\ 200 \end{bmatrix}$<br>$= \begin{bmatrix} 2 & -\frac{1}{25} \\ -1 & \frac{1}{25} \end{bmatrix} \times \begin{bmatrix} 6 \\ 200 \end{bmatrix}$<br>$= \begin{bmatrix} (2\times 6) + (-\frac{1}{25}\times 200) \\ (-1\times 6) + (\frac{1}{25}\times 200) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ |

(It is important to note that if we are trying to solve a system of equations and the determinant turns out to be **0**, that system either has an **infinite number of solutions**, or **no solution**.)

# Solving Matrix Equations

Here are a couple more types of matrices problems you might see:

### Matrix Multiplication Problem

Let $P = \begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix}$. (a) Find $2P$, (b) Find $P^2$, (c) Find $Q$ when $P \times Q = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

**Solutions:**

(a) When we multiply a matrix by a scalar (number), we just multiply all elements in the matrix by that number. So

$$2P = 2\begin{bmatrix} 2\times 4 & 2\times -6 \\ 2\times -2 & 2\times 8 \end{bmatrix} = \begin{bmatrix} 8 & -12 \\ -4 & 16 \end{bmatrix}.$$

(b) When we square ***P***, we just multiply it by itself. Let's do this "by hand":

$$P^2 = \begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix} \times \begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} (4\times 4)+(-6\times -2) & (4\times -6)+(-6\times 8) \\ (-2\times 4)+(8\times -2) & (-2\times -6)+(8\times 8) \end{bmatrix} = \begin{bmatrix} 28 & -72 \\ -24 & 76 \end{bmatrix}$$

(c) Since $\begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix} \times Q = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, we have $Q = \begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix}^{-1} \times \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ (sort of like when we're solving a system). Let's use our calculator to put P in [A] and $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ in [B]. Then $[A]^{-1}\times[B] = \begin{bmatrix} 2 \\ .5 \end{bmatrix}$.

### Matrix Equation Problem:

This one's a little trickier, since it doesn't really look like a systems problem, but you solve it the same way:

Solve the matrix equation for X (X will be a matrix):

$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} X - \begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix}$$

**Solution:**

Let's add the second matrix to both sides, to get X and its coefficient matrix alone by themselves. Then we'll "divide" by the matrix in front of X. Watch the order when we multiply by the inverse (matrix multiplication is not commutative), and thank goodness for the calculator!

$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} X - \begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix} + \begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} X = \begin{bmatrix} 9 & -6 \\ -4 & 11 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 9 & -6 \\ -4 & 11 \end{bmatrix} = \begin{bmatrix} \frac{24}{11} & \frac{9}{11} \\ \frac{17}{11} & -\frac{28}{11} \end{bmatrix}$$

Watch order!

We can check it back: $\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} \frac{24}{11} & \frac{9}{11} \\ \frac{17}{11} & -\frac{28}{11} \end{bmatrix} - \begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix}$? It works!

# Matrix Equations Worksheet

Show work on separate paper

1. Given the matrices:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Solve the matrix equation:

$$AX = B$$

2. Given the matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solve the matrix equation:

$$XA + B = C$$

3. Given the matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solve for the matrix equations:

a. $XA = B + I$ (I is the identity matrix)

b. $AX + B = C$

c. $XA + B = 2C$

d. $AX + BX = C$

e. $XAB - XC = 2C$